Math 323 - Formal Mathematical Reasoning and Writing

Problem Session

Wednesday, 2/25/15

Recall the following theorem:

Theorem 1 *If $a, b \in F$ an ordered field with $a < b$, then there is an element $r \in F$ such that $a < r < b$. In fact, $r = \frac{a+b}{2}$ is one example.*

Write solutions to the following problems. Use as much detail as you would on a homework assignment.

The idea of the proof in Problem 2(a) will be useful for #2 on the Problem Set. Problem 3 might be useful for #3 on the Problem Set. Problem 4 might (provided you think far enough along in the problem...) be useful for #5 on the Problem Set.

1. Which of the following are well defined?
   (a) The minimum of the set S.
   (b) The lower bound of the set S.
   (c) A minimum of the set S.
   (d) A lower bound of the set S.
2. Let $a, b \in \mathbb{Q}$ with $a > 1$ and $b > 0$.
   (a) Prove that there exists a rational number x such that $ax = |x - b|$.
   (b) How many elements are in the set $S = \{x \in F \mid ax = |x - b|\}$?
3. Let K be an ordered field, and let $a, b \in K$ with $a < b$. Prove that there exist numbers $r_1, r_2 \in K$ with $a < r_1 < r_2 < b$.
4. Suppose that $a, b \in \mathbb{Q}$. Prove that if $a^2 = 2b^2$, then $a = b = 0$.
5. **Mathematical Virtue** Recall that

$$\mathbb{Q}(\sqrt{2}) = \{a + b\phi \mid a, b \in \mathbb{Q}\},$$

   and that the order $\lessdot$ is defined by

$$s + t\phi \lessdot u + v\phi \quad \Longleftrightarrow \quad |s - u|(s - u) < 2|v - t|(v - t).$$

   Your task in the mathematical virtue problem is to show that $\lessdot$ satisfies trichotomy.

$\LaTeX$ tip of the week!

If you want to change the numbering in an `enumerate` list, you can do the following:

```
\begin{enumerate}
    \item[0)] Question zero!
    \item[1)] Question 1!
    \item[\smiley] A happy item!
\end{enumerate}
```

0) Question zero!

1) Question 1!

☺ A happy item!

Ok, I can't help myself. **Bonus Tip!** Using `\mid` can make spacing a bit nicer.

1. `\{ a + b\phi | a,b \in \mathbb{Q} \}` gives us:

$$\{a + b\phi | a, b \in \mathbb{Q}\}$$

2. `\{ a + b\phi \mid a,b \in \mathbb{Q} \}` gives us:

$$\{a + b\phi \mid a, b \in \mathbb{Q}\}$$